Rosenthal compact spaces

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Arbitrary sets	$\leftrightarrow$	Analytic sets
Arbitrary compacta	$\leftrightarrow$	Separable Rosenthal compacta

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Separable Rosenthal compact spaces

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- K is *something* $\iff K \supset C_1$?

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"Double G_δ-point"

$\exists \mathscr{U}$ countable family of open sets
$\forall y \neq z \neq x \ \exists W_x, W_y, W_z \in \mathscr{U} \ \ W_x \cap W_y \cap W_z = \emptyset.$

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A more set-theoretic *version* of this problem

If $\mathscr{B}$ is a Borel subalgebra of $\mathscr{P}(\omega)$ that does not contain $\mathscr{P}(\omega)$, is $\mathscr{B}$ σ-scattered in the pointwise topology?

Functions with countably many discontinuities

Partial answer (Haydon, Moltó, Orihuela)

Yes, if K is made of functions with countably many discontinuities.

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$$CD \subset R$$

Separable compact spaces:

R made of Baire_1 functions on Polish space

CD made of functions with $\aleph_0$ discontinuities

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Yes, if K is made of functions with countably many discontinuities.

$$CD \subset RK \subset R$$

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- R made of Baire_1 functions on Polish space
- RK made of Baire_1 functions on compact metric
- CD made of functions with $\aleph_0$ discontinuities

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$$CD \subset RK \subset R$$

Separable compact spaces:

R made of Baire_1 functions on Polish space

RK made of Baire_1 functions on compact metric

CD made of functions with $\aleph_0$ discontinuities

Pol, Marciszewski-Pol: $RK \neq R$ A.-Todorcevic: $CD \neq RK$

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Proof

$K \subset \{f : X \longrightarrow \mathbb{R}\}$
$D \subset X$ countabe dense, and $r : K \longrightarrow \mathbb{R}^D$ the restriction.

Example: Take K the space of all functions $f : [0,1]^2 \longrightarrow \{0,1\}$ lexicographically non-decreasing.

A problem

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Is every separable Rosenthal compactum a continuous image of a *CD* space?

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We know that this is the case for (separable supplementations of) lexicographically increasing functions $[0,1]^n \longrightarrow \mathbb{R}$....
but what about larger ordinals?